Lecture notes

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12 September 2017

- Continouse time fourie transfrom
- Rep + Continouse time fuirie series
- Energi + Rep

Repition

$x : \mathbb{R} -> \mathbb{C}, \int_{-\infty}^{\infty} |x(t)| dt < \infty$

Så defineras dess fourie transform $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}, \omega \in \mathbb{R}$

Kan referera till ω som frekvens

Egenskaper för fourie transfrom

$$x = \alpha_1 x_1 + \alpha_2 x_2, \rightarrow X = \alpha_1 X_1 + \alpha_2 X_2$$

X bestäms entydligt av x. Om $\int_{-\infty}^{\infty} |X(j\omega)| d\omega < \infty$ så gäller

$$x(t) = \frac{1}{2\pi} \int_{-infty}^{\infty} X(j\omega) e^{j\omega t} d\omega, \forall t \in \mathbb{R}$$

Viktiga transformer

- $y = h * x \iff Y = H \cdot X$
- $z(t) = x(t - t_0) = Z(j\omega) = e^{-j\omega t} X(j\omega)$
- $x(t) = t^n \cdot e^{-at} u(t), a > 0 \iff X(j\omega) = \frac{(n-1)!}{(a+j\omega)^n}$

Exempel

- Stabilt LTI-system $y = h * x$. Antag att

  $$x(t) = e^{-2t} u(t)$$

  ger

  $$y(t) = 2te^{-3t} u(t)$$

  Om $x(t) = e^{-3t} u(t)$ Vad blir y?

- Lösning:

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

I vårt fall: $X(j\omega) = \dfrac{1}{2 + j\omega}$

$$Y(j\omega) = \frac{2}{(3+j\omega)^3} \to H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\frac{2}{(3+j\omega)^3}}{\frac{1}{2+j\omega}}$$

$$=2 \cdot \frac{2+j\omega}{(3+j\omega)^2} = 2 \cdot \frac{3+j\omega-1}{(3+j\omega)^2} = 2 \cdot \left(\frac{1}{3+j\omega} - \frac{1}{(3+j\omega)^2}\right) = H_1(j\omega) - H_2(j\omega)$$

Bestäm h

$$H = 2(H_1 - H_2) \to h = 2(h_1 - h_2)$$
$$h_1 = e^{-3t}u(t), (3, n = 1, a = 3)$$
$$h_2 = te^{-3t}u(t), (3, n = 2, a = 3)$$
$$h(t) = 2e^{-3t}(1 - t)u(t)$$

Om nu

$$x(t) = e^{-3t}u(t) \to X(jw) = \frac{1}{3 + j\omega}$$

$$Y(jw) = H(j\omega) \cdot \frac{1}{3+j\omega} = \frac{2}{(3+j\omega)^2} - \frac{2}{(3+j\omega)^3} = Y_1(j\omega) - Y_2(j\omega)$$

$$\to y(t) = y_1(t) - y_2(t)$$
$$y_1(t) = 2te^{-3t}u(t)$$
$$y_2(t) = t^2e^{-3t}u(t), (3, n = 3, a = 3)$$
$$\to y(t) = 2t^{-3t}u(t) - t^2e{-}3tu(t) = te^{-3t}(2 - t)u(t)$$

Fourieserier

Stabilt LTI-system $y = h * x$

$$\int_{-infty}^{\infty} |h(t)|dt < \infty$$

OBS: Om x är T-periodisk alltså $x(t + T) = x(t), \forall t \in \mathbb{R}$. så blir också utsignalen T-periodisk.

$$y(t - T) = \int_{-\infty}^{\infty} h(\tau)x(t + T - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau = (h * x)(t) = y(t)$$

Kom ihåg $x_\omega(t) = e^{j\omega t} \to y(t) = H(j\omega)x_\omega(t)$.

$$x_w \text{ T-periodisk} \iff \omega = \omega_k = \frac{2\pi k}{T}, \text{k heltal}$$

$$x_{\omega_k}(t + T) = e^{j\omega_k(t+T)} = e^{j\omega_k t} \cdot e^{j\omega_k T} = e^{j\omega_k t} = x_\omega(t)$$

Defintion: Om $x : \mathbb{R} \to \mathbb{C}$ T-periodisk och

$$\int_0^T |x(t)| dt < \infty$$

så ddefineras dess fourierseriekoeff (c_k) enligt

$$c_k(x) = \frac{1}{T}\int_0^T x(t)e^{-j\omega_k t} dt, \omega_k = \frac{2\pi k}{T}$$

Egenskaper

- $x = \alpha_1 x_1 + \alpha_2 x_2 (x_1, x_2$ T-periodiska$) \to c_k(x) = \alpha_1 c_k(x_1) + \alpha_2 c_k(x_2)$.
- x bestämms entydligt av c_k. Explicit:

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty \to x(t) = \sum_{k=-\infty}^{\infty} c_k(x) e^{j\omega_k t} \forall t \in \mathbb{R}$$

  Exempel

  - $x(t) = cos(2t)$ Vad är c_k? $[T = \pi]$
  - Lösning: Alternativ 1

$$c_k(x) = \frac{1}{\pi}\int_0^{\pi} cos(2t) e^{-j\frac{2\pi k}{\pi}t} dt = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}(e^{2jt} + e^{-2jt}) \cdot e^{-2jkt} dt$$

$$\frac{1}{2\pi}\int_0^{\pi} e^{2jt(1-k)} dt + \frac{1}{2\pi}\int_0^{\pi} e^{-j2t(1+k)} dt =$$

$$= \frac{1}{2\pi}\left[\frac{e^{2jt(1-k)}}{2j(1-k)}\right]_{t=0}^{\pi} + \frac{1}{2\pi}\left[\frac{-e^{-j2t(1+k)}}{j2(1+k)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{1-1}{2j(1-k)}\right) = 0, k \neq 1, -1$$

$$c_1(x) = 1/2, c_{-1}(x) 1/2$$

$$c_k(x) = \begin{cases} 0, k \neq 1, -1 \\ 1/2, k = 1, -1 \end{cases}$$

$$x(t) = c_{-1}(x) \cdot e^{j\omega_{-1}t} + c_1(x) \cdot e^{j\omega_1 t} = \frac{1}{2}(e^{-j2t} + e^{j2t}) = cos2t$$

  Alternative 2:

$$x(t) = \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t} = c_{-1}(x)e^{j\omega_{-1}t} + c_1(x)e^{j\omega_1 t}$$

  Exempel: Bestämm c_k om $x(t) = sint + cos3t$

$$\begin{cases} x(t) = \frac{1}{2j}(e^{jt} - e^{-jt}) + \frac{1}{2}(e^{3jt} + e^{-3jt}) \\ \omega_k = \frac{2\pi k}{2\pi} = k \end{cases} \to c_k(x) = \begin{cases} 1/2j, k = 1 \\ -1/2j, k = -1 \\ 1/2, K = \pm 3 \\ 0 \end{cases}$$

Fundamentalt samband

Om x är T-periodisk så gäller,

$$c_k(y) = H(j\omega)c_k(x)\forall k, \omega_k = \frac{2\pi k}{T}$$

Exempel

Antag

$$x(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k^2} coskt$$

$$H(j\omega) = \begin{cases} (1-2|\omega|)^{17}, |\omega| < 1/2 \\ 0, \text{övrigt} \end{cases}$$

Bestäm y. Lösning $[T = 2\pi.\omega_k = k]$

$$c_k(y) = H(j\omega_k) \cdot c_k(x) = \begin{cases} 0, k \neq 0 \\ c_0(x) = 1/2, k = 0 \end{cases}$$

$$\rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k(y) e^{jkt} = \frac{1}{2} \forall t$$